Towards QCD thermodynamics using exact chiral symmetry on lattice

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Work done with:

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Outline

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Introduction

- Fermions with exact chiral symmetry: Domain wall fermions (Kaplan ′92) Overlap fermions (Narayanan, Neuberger ′95)

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- Important for studies of chiral symmetry restoration in QCD.

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- Fermions with exact chiral symmetry: Domain wall fermions (Kaplan '92) Overlap fermions (Narayanan, Neuberger '95)
- Important for studies of chiral symmetry restoration in QCD.
- The systematic study of the thermodynamics of free gases of such fermions will give us an idea of the optimum choice of lattice parameters for full QCD simulations.

The topics addressed

- What values of M give us the best possible description of thermodynamics of free fermions? Does this choice depend upon the aspect ratio chosen?
  We note that $0 < M < 2$ for the simulation of one-flavour fermion on lattice.
  Previously,chosen values of M
  $M = 1.0$ for free fermion simulations. $M = 1.7 - 1.9$ for QCD simulations.

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- For the Domain wall fermions:what is the optimum length L_5 in the fifth dimension required for obtaining ideal 4D Fermi gas results.
- How can Overlap formalism be used in presence of a finite chemical potential?
  I would like to make some comments on the work initiated by Gattringer and Liptak(2007) .

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The Energy density of Overlap fermions

- The Overlap operator for massless fermions

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- The energy density on the lattice is

$$\epsilon = -\frac{1}{N^3 a^3 N_T} \left(\frac{\partial ln \det D_{ov}}{\partial a_4} \right)_{a=const}$$

$$\epsilon = \frac{2}{N^3 a^4 N_T} \sum_{p_j, p_4} \frac{h^2(1 - cos(ap_4)) + h_5 h_4^2}{h^2(h^2 + h_5^2)} (\sqrt{h^2 + h_5^2} + h_5)$$

where

$$\begin{aligned} h_5 &= M - \sum_{j=1}^{4} (1 - cos(ap_j)) \\ h_j &= -\sin(ap_j), j = 1, 2, 3, 4 \\ h^2 &= h_1^2 + h_2^2 + h_3^2 + h_4^2 \end{aligned}$$

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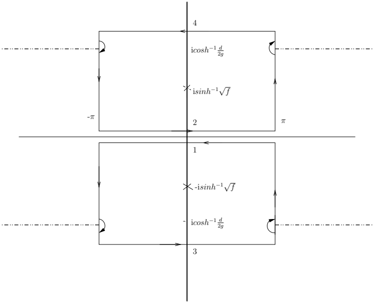

Energy Density of Overlap fermions

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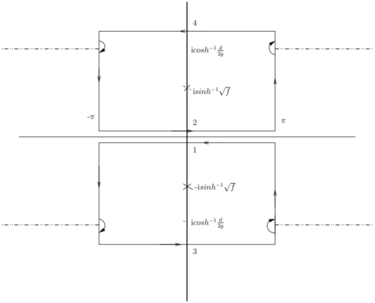

- The exact expression for the energy density can be analytically computed:

$$
\begin{aligned}
(\epsilon-\epsilon_0)a^4 &= \frac{2}{N^3}\sum_{p_j}\Big(\frac{\sqrt{f}(1+f+2g\sqrt{1+f}+g^2)}{\sqrt{1+f}(d+2g\sqrt{1+f})} \\
&+ \frac{\sqrt{f}(\sqrt{1+f}+g)}{\sqrt{1+f}\sqrt{d+2g\sqrt{1+f}}}\Big)\frac{1}{e^{N_T sinh^{-1}\sqrt{f}}+1}
\end{aligned}
$$

This expression gives the correct continuum limit.

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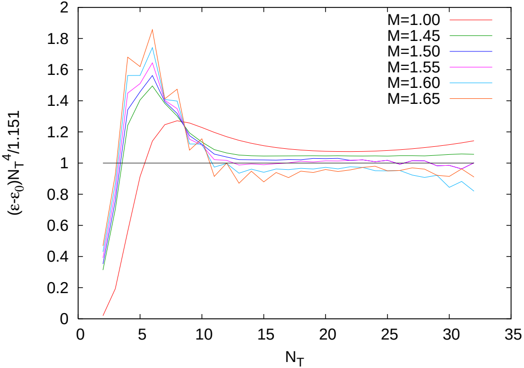
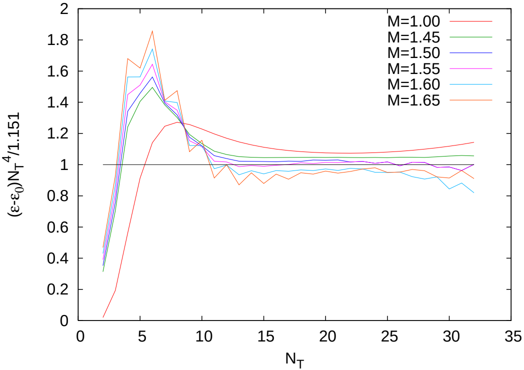
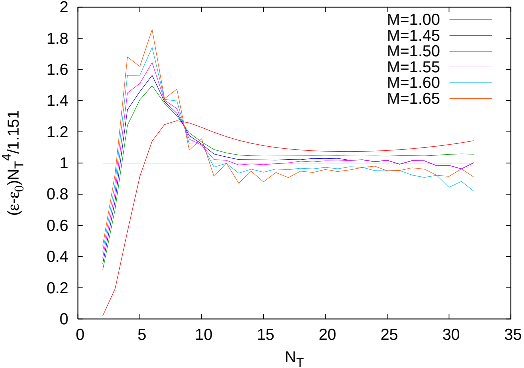


- The plots are for $\zeta = \frac{N_s}{N_T} = 5$.
  .The continuum value of the energy density for fermion-anti-fermions is given by $\epsilon - \epsilon_0 = \frac{7\pi^2 T^4}{60} = 1.151\,T^4$.
  The *continuum* result is obtained for $N_T > 12$.

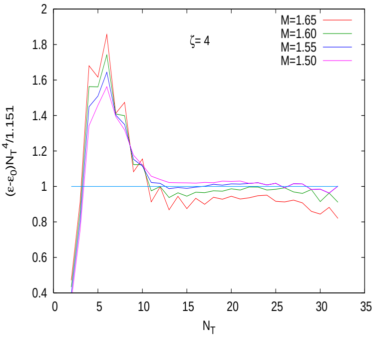


- M dependence exist on lattice even for lattice sizes $N_T = 32$. This M dependence should go away for very large lattices. $M = 1.55$ is the suitable choice for getting results that are close to continuum result and with reasonable lattice size.



- The lattice energy density for $M = 1.0$ never comes near the continuum result even at $N_T = 32$.

Results : Overlap fermions



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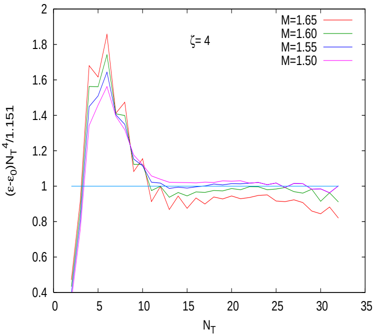

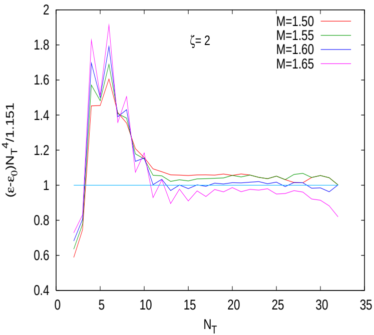

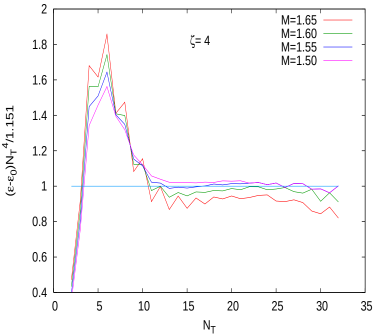

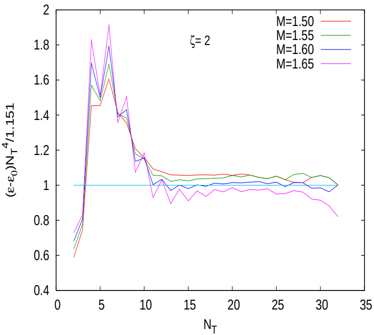

- For continuum results
  $N_{Tmin} = 12$ for $\zeta = 4$
  $N_{Tmin} = 15$ for $\zeta = 2$

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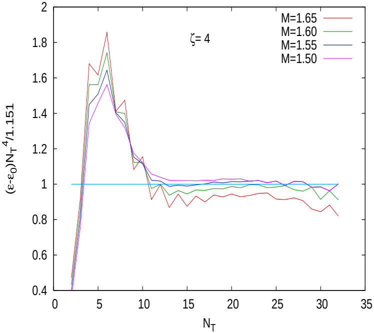

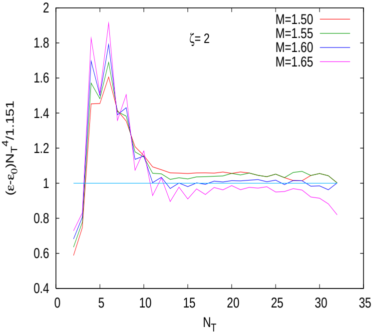

- For continuum results
  $N_{Tmin} = 12$ for $\zeta = 4$
  $N_{Tmin} = 15$ for $\zeta = 2$
- The optimum M also varies:
  $M = 1.55$ for $\zeta = 4$
  $M = 1.60$ for $\zeta = 2$.

Energy density: Domain wall fermions

- The Domain wall effective operator for massless fermions:

$$D_{DW} = 1 + \gamma^5 \frac{1 - T^{N_5}}{1 + T^{N_5}}$$

where

$$T = \frac{1 - a_5 H_t}{1 + a_5 H_t}$$
$$H_t = \gamma^5 D_W (2 + a_5 D_W)^{-1}$$

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- When $N_5 \to \infty$, $a_5 \to 0 \implies L_5 = N_5 a_5 = finite$
  The energy density for Domain wall fermion:
  $\varepsilon a^4 =$
  $$\sum_{p_j, p_4} \frac{4\sinh[\frac{sL_5}{2}]((-h_4 h_5 \alpha + h^2\gamma)\cosh[\frac{sL_5}{2}] + (h_4 h_5 \alpha + 2h_5^2(h_5 p + \gamma) + h^2(2h_5 p + \gamma))}{sN^3 N_T (h^2 + (s^2 + h_5^2)\cosh[2sL_5]}$$
  $$\frac{\cosh(\frac{3sL_5}{2}) - 2s(h_5^2 p + h_5\gamma + (h^2 p + h_4\alpha + 2h_5(h_5 p + \gamma))\cosh[sL_5])\sinh[\frac{s\,L_5}{2}])}{-4h_5\ s\cosh[sL_5]\sinh[sL_5])}$$

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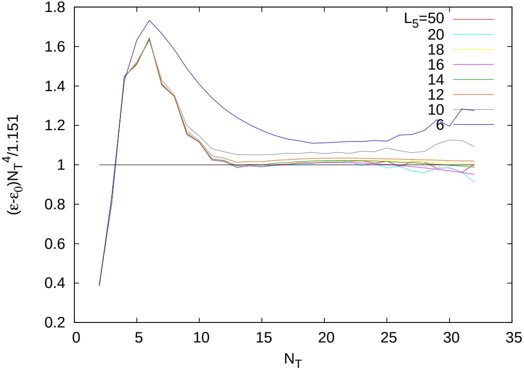
Energy density: Domain wall fermions



- The energy density matches with the continuum limit for $L_5 > 14$ for $\zeta = 4$

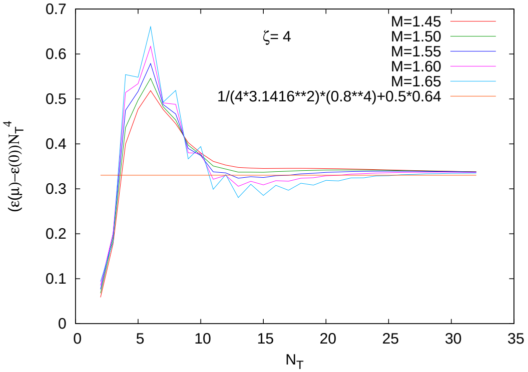
Outline

Overlap fermions in presence of chemical potential

- In presence of μ: The energy density on the lattice can be computed exactly by converting the frequency sum to contour integral. The expression for energy density in presence of μ is

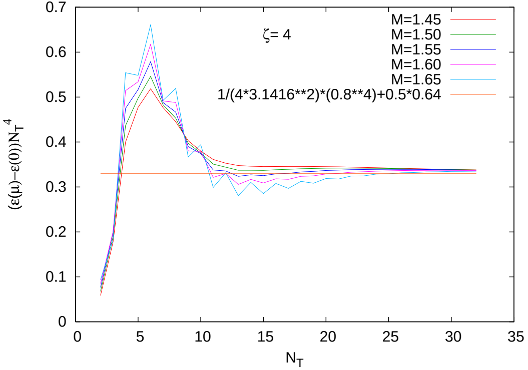
$$
\begin{aligned}
\epsilon - \epsilon_0 &= \frac{2}{N_T N^3 a^4} \sum_{\omega_n, p_j} F(\omega_n - i\mu) \\
&= \frac{1}{N^3 a^4} \sum_{p_j} [\frac{\sqrt{f}(1 + f + 2g\sqrt{1+f} + g^2)}{\sqrt{1+f}(d + 2g\sqrt{1+f})} \\
&+ \frac{\sqrt{f}(\sqrt{1+f} + g)}{\sqrt{1+f}\sqrt{d + 2g\sqrt{1+f}}}] \frac{1}{e^{N_T(sinh^{-1}\sqrt{f} + \mu)} + 1} \\
&+ \frac{1}{N^3 a^4} \sum_{p_j} [\frac{\sqrt{f}(1 + f + 2g\sqrt{1+f} + g^2)}{\sqrt{1+f}(d + 2g\sqrt{1+f})} \\
&+ \frac{\sqrt{f}(\sqrt{1+f} + g)}{\sqrt{1+f}\sqrt{d + 2g\sqrt{1+f}}}] \frac{1}{e^{N_T(sinh^{-1}\sqrt{f} - \mu)} + 1}
\end{aligned}
$$

Overlap fermions in presence of chemical potential



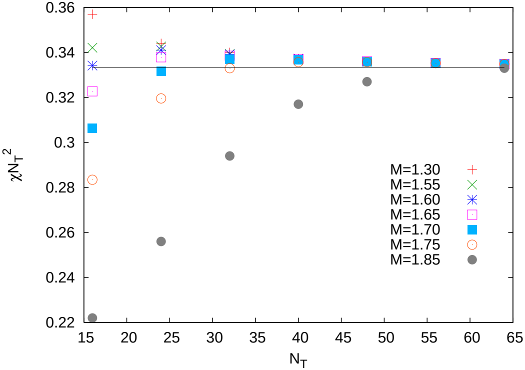
- In presence of μ: The energy density $\longrightarrow$ $\frac{\epsilon(\mu)-\epsilon(0)}{T^4} = \frac{\mu^4}{4\pi^2 T^4} + \frac{\mu^2}{2T^2} + \frac{\mu^2}{a^2 T^4} = \frac{r^4}{4\pi^2} + \frac{r^2}{2} + \frac{r^2}{a^2 T^2}$ where ratio $r = \frac{\mu}{T}$ is kept fixed. The sign function is not always defined. ▸ more

Overlap fermions in presence of chemical potential



- lattice divergences $\rightarrow$ There are no $\frac{\mu^2}{a^2}$ effects seen for $N_T > 15$ for different aspect ratios. $\rightarrow$ approach to the continuum limit.
  Weak M dependence vanishing at $N_T \rightarrow 32$..

Overlap fermions in presence of chemical potential



- The succeptibility is known to be

$$\frac{\chi(\mu)}{T^2} = \frac{\mu^2}{\pi^2 T^2} + \frac{1}{3}$$

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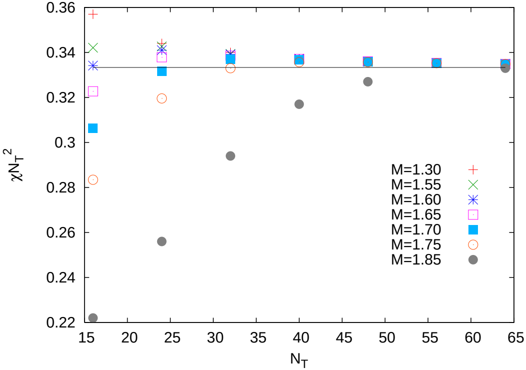

- The succeptibility is known to be

$$\frac{\chi(\mu)}{T^2} = \frac{\mu^2}{\pi^2 T^2} + \frac{1}{3}$$

- The χ at zero chemical potential approaches the continuum value for very large lattice size. The value of $M = 1.6$ could give us $\chi(0)$ with a comparatively smaller lattice.

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Conclusions

- The energy density of the fermions with chiral symmetry on lattice depend on the parameter M. This dependence should go away in the continuum $\rightarrow$ the approach is very slow. Our studies show that $M = 1.5 - 1.6$ is the optimum range for numerical simulations of energy density of overlap fermions both in the presence and absence of a finite chemical potential.
- For Domain wall fermions the thermodynamic limit is reached for the same values of M but for $L_5 > 14$.
- The optimum range of the M determined in the case for the free fermions can be made use of in the interacting cases also. The presence of gauge fields may smoothen out fluctuations and allow a larger range of M to be used but M values near 1 are not desirable for practical purposes.
- The succeptibility results also indicate $M = 1.6$ would be an optimum choice.

Acknowledgements

- I would like to acknowledge the Council for Scientific and Industrial Research for the SPM fellowship.

Energy Density : Overlap fermions

- The energy density on lattice

$$\begin{aligned}\epsilon &= -\frac{1}{N^3 a^3 N_T}\left(\frac{\partial ln \det D}{\partial a_4}\right)_{a=const} \\ &= -\frac{1}{N^3 a^3 N_T}\left(\frac{\partial ln(\prod_n \lambda_n)}{\partial a_4}\right)_{a=const} \\ &= -\frac{2}{N^3 a^3 N_T}\sum_{\pm}\left(\frac{\partial ln\lambda_\pm}{\partial a_4}\right)_{a=const}\end{aligned}$$

The derivatives of h's are given by $\frac{\partial h_4}{\partial a_4} = -\frac{h_4}{a_4}$
$\frac{\partial h_5}{\partial a_4} = \frac{a}{a_4^2}(1-\cos(a_4 p_4))$
Also we use the following definitions $h_5 = M - \sum_{j=1}^{4}(1 - cos(ap_j))$
$h_j = -\sin(ap_j), \mathrm{j} = 1, 2, 3$
$h_4 = -\frac{a}{a_4}\sin(a_4 p_4)$
$h^2 = h_1^2 + h_2^2 + h_3^2 + h_4^2 \; s^2 = h_1^2 + h_2^2 + h_3^2 + h_4^2 + h5^2$

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Energy Density: Overlap fermions

- In the energy density expression on lattice the following quantities are used

$$\begin{aligned}
b &= \cos(ap_1) + \cos(ap_2) + \cos(ap_3) \\
g &= M - 4 + b \\
f &= h_1^2 + h_2^2 + h_3^2 \\
c &= \sum_{i<j;j<4} 2\cos(ap_i)\cos(ap_j) \\
d &= 4 + (M-4)^2 + 2(M-4)b + c
\end{aligned}$$

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Energy Density: Domain Wall fermions

- For the energy density expression on lattice the following quantities are used:
  $D_{DW} = \frac{1}{a}[-i\Sigma_{i=1}^{3} h_i \gamma^i - ih_4\gamma^4 - h_5]$
  $\alpha = -h_4$
  $\gamma = 1 - cos((2m+1)\pi/N_T)$
  $p = \frac{(-\sin^2 ap_4 + h_5(1-\cos ap_4)(-\tanh L_s s + L_s s \,\mathrm{sech}^2 L_s s)}{s^2 \tanh L_s s}$

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Energy Density: Domain Wall fermions

- For the energy density expression on lattice the following quantities are used:
  $D_{DW} = \frac{1}{a}[-i\Sigma_{i=1}^{3} h_i \gamma^i - i h_4 \gamma^4 - h_5]$
  $\alpha = -h_4$
  $\gamma = 1 - cos((2m+1)\pi/N_T)$
  $p = \frac{(-\sin^2 ap_4 + h_5(1-\cos ap_4)(-\tanh L_s s + L_s s \operatorname{sech}^2 L_s s)}{s^2 \tanh L_s s}$
- The energy density has been solved analytically by us. The bulk modes contribute to the energy density even for very large lattice size. Only the fermions bound to the domain wall give us the $4D$ continuum result.

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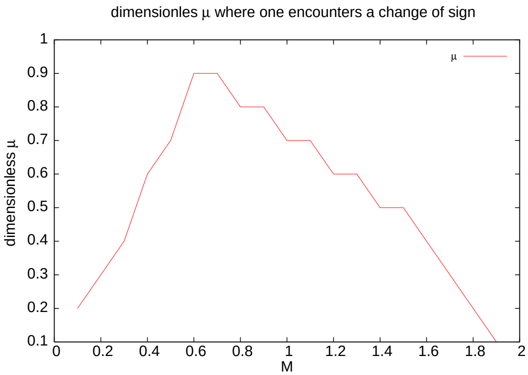
Overlap fermions in presence of chemical potential



- The maximum allowed value of μ beyond which the argument of sign function is purely imaginary depend on M.

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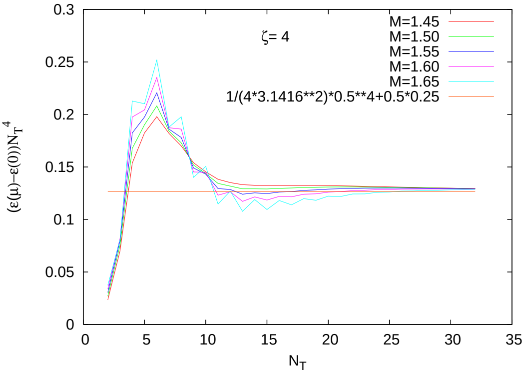
Overlap fermions in presence of chemical potential



Succeptibility of Overlap fermions